FIBONACCI SPIRAL IN THE CARTESIAN COORDINATE SYSTEM: IMAGINATION FROM GRAPH TO INFINITE LENGTH AND GOLDEN RATIO

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Abstract

This project originated from an interest in the Fibonacci sequence (1,1,2,3,5, 8,...), a simple sequence possessing fascinating mathematical characteristics and appearing widely in nature, such as in the patterns of flower petals, tropical storms, and galaxies. The authors began by imagining the mapping of Fibonacci numbers as points on the axes of a Cartesian coordinate system, plotted in an infinitely spiraling sequence. By continuously connecting each pair of points with line segments, a geometric shape resembling a spiral was formed, referred to in this project as the "Fibonacci Spiral on the Cartesian Plane." From this figure, the authors observed patterns and formulated conjectures, leading to the proof of four interesting theorems: 1) the length of the n-th line segment in the spiral can be determined from the square root of a specific Fibonacci number; 2) the ratio of the lengths of adjacent line segments converges to the Golden Ratio (φ); 3) the ratio of the areas of adjacent triangles within the spiral converges to the square of the Golden Ratio; and 4) adjacent line segments are perpendicular to each other, which implies that segments separated by two orders are parallel. This project serves as a clear example that mathematics is not merely a subject in textbooks but the language nature uses to design itself. From ordinary straight lines connecting points, the project reveals the profundity of infinite geometry through observation, questioning, and the search for hidden meaning in simple things. Beyond theoretical understanding, the developed model could support future applications, such as modeling the eyes of storms or galaxies, while serving as a comprehensive learning platform that encompasses observation, analysis, proof, creative thinking, and the use of technology to visualize results.

Keywords: Fibonacci Numbers, Golden Ratio, Fibonacci Spiral, Infinite Length

Introduction

The organizer of this project began with an interest in "Fibonacci Numbers" which are numbers created simply but conceal a profound mystery. The sequence starts with 1 and 1, and the next number is obtained by adding the two previous numbers, becoming an infinite sequence that appears abundantly in nature. Examples include the arrangement of flower petals, the patterns of seashells, the spirals of storms, and even the structure of galaxies. Upon close observation, one finds that Fibonacci numbers are embedded everywhere in the nature around us, as if nature were using mathematics to communicate with us. This initial understanding excited me and drove the desire for an in-depth study to discover how this ordinary sequence of numbers could be further developed in mathematics.

The inspiration for this project originated from a simple question: "What would happen if we took Fibonacci numbers to define points on a Cartesian coordinate system, arranging the positions in a counterclockwise or clockwise rotation on the X and Y axes, and then connected those points with line segments?"

This question seemed like mere doodling at first. However, when I attempted it, I discovered that the geometric shape formed by connecting the points based on Fibonacci numbers turned into a shape resembling a "spiral" that swirls infinitely. Visually, it possesses beautiful proportions and imagining it expanding infinitely is indescribably exciting. I call this geometric shape the **"Fibonacci Spiral in the Cartesian Coordinate System"** and used it as the starting point to study the mathematical relationships hidden within. This was to answer the internal question of why this figure is so striking—perhaps it holds a relationship with the Golden Ratio like the Mona Lisa or possesses other interesting mathematical properties yet to be discovered. This study integrates mathematical knowledge from various branches, including sequences, number theory, limits, and geometry, to search for exciting new knowledge.

Upon serious consideration, I found that each line segment connecting the Fibonacci points in this spiral has a geometric relationship with the Pythagorean theorem. Furthermore, hypotheses regarding the length, area, and angles occurring in each section of the spiral could be systematically established. I began testing and proving these observations using high school-level mathematics and was able to formulate several theorems. Specifically, the relationship between the length of line segments and Fibonacci numbers can be summarized into a general formula for easy calculation. Additionally, it was found that the ratios of lengths or areas in the spiral tend to converge to the **"Golden Ratio,"** a constant significant in mathematics, art, architecture, and nature. Beyond theoretical proof, I developed a simple computer program to assist in calculating the relevant lengths and ratios, which helped reduce complexity and added interest to the project. Creating this program allowed me to practice applying mathematics to technology and realize that mathematics is not a distant subject but can be used to create truly useful tools.

Therefore, this project is not merely about drawing pictures or calculating ordinary numbers, but about integrating mathematical knowledge with observation, hypothesis setting, proof,application, and systematic communication of ideas. I view this project as an opportunity for a learner like me to practice deep analytical thinking, learn to apply mathematics in the real world, and open a new perspective that makes mathematics fun, challenging, and meaningful. I am confident that this project will serve as an example of learning that starts with a small curiosity and expands into a deeper understanding. It aims to inspire a new generation of learners to see that mathematics is not just a subject in a textbook but a language that can describe the beauty of this world amazingly. History has shown that mathematical discoveries appearing only as theoretical beauty at first can become the foundation of technologies that change the world later—just as the Fibonacci sequence discovered in the 13th century has become the inspiration for numerous modern innovations, from art and architecture to the design of computer algorithms.

Methodology

The implementation of the project "Fibonacci Spiral in the Cartesian Coordinate System: Imagination from Graph to Infinite Length and Golden Ratio" is a mathematical study that integrates observation, the formulation of conjectures, theoretical proof, and the application of technology.

Result

If we place Fibonacci numbers on a Cartesian coordinate system without regard to the sign on the X and Y axes, rotating clockwise (or counterclockwise), we find that the line segments connecting adjacent Fibonacci numbers form a shape resembling a spiral that swirls infinitely. In this project, we will call this geometric figure the "Fibonacci Spiral in the Cartesian Coordinate System" as shown in Figure 1.

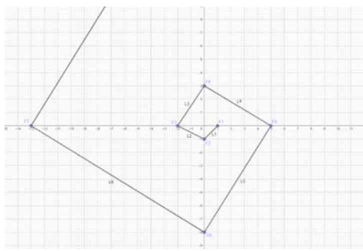

Figure 1 Fibonacci Spiral in the Cartesian

In this context, we define L_n as the line segment connecting point F_n and point F_{n+1}, and $|L_n|$ represents the length of L_n. By observing the Fibonacci spiral, we find that L_1 is the hypotenuse of a right-angled triangle with sides F_1 and F_2 and F_2 is the hypotenuse of a right-angled triangle with sides F_2 and F_3. This continues indefinitely. By the Pythagorean theorem, we get $|L_1|^2 = F_1^2 + F_2^2 = 2 = F_3$, $|L_2|^2 = F_2^2 + F_3^2 = 5 = F_5$ and so on.

We observe that $|L_n|^2 = F_{2n+1}$ for all $n \geq 1$. Therefore, if we can prove that this observation is true, we will obtain a general formula for finding the length of $|L_n|$ easily by considering the sequence of Fibonacci numbers. This leads to Conjecture 1 as follows:

Conjecture 1 $|L_n|^2 = F_{2n+1}$ for all $n \geq 1$.

Proof: Since L_n is the hypotenuse of a right-angled triangle with legs F_n and F_{n+1}, Therefore,

$$|L_n|^2 = F_n^2 + F_{n+1}^2 \qquad (1)$$

From fundamental knowledge (Theorem 2.1) $F_{2n-1} = F_n^2 + F_{n-1}^2$, substituting n with n+1, we get that $F_{2(n+1)-1} = F_{n+1}^2 + F_{(n+1)-1}^2$ Therefore,

$$F_{2n+1} = F_{n+1}^2 + F_n^2 \qquad (2)$$

From equations (1) and (2), we get: $|L_n|^2 = F_{2n+1}$

So, we obtain the following theorem:

Theorem 1 The length of L_n is equal to the square root of the (2n+1)-th Fibonacci number. That is: $|L_n| = \sqrt{F_{2n+1}}$ for all $n \geq 1$.

Example $|L_3| = \sqrt{F_{2(3)+1}} = \sqrt{F_7} = \sqrt{13}$, $|L_4| = \sqrt{F_{2(4)+1}} = \sqrt{F_9} = \sqrt{34}$

It can be observed that finding $|L_n|$ using Theorem 4.1 requires the (2n+1)-th Fibonacci number. For example, to find $|L_{50}|$, we need to find F_{101}, which is the 101st Fibonacci number. This task is time-consuming using the recursive relation from the definition of Fibonacci numbers. The following lemma will help us find $|L_n|$ more conveniently using a Closed-form formula.

Lemma 1 $|L_n| = \left(\varphi^{2n+1} - (1-\varphi)^{2n+1} / \sqrt{5}\right)^{\frac{1}{2}}$

Proof: From fundamental knowledge (Binet's Formula), we have: $F_n = (\varphi^n - (1-\varphi)^n) / \sqrt{5}$

Therefore, substituting n with 2n+1, we get that

$$|L_n| = \sqrt{F_{2n+1}} = \left(\varphi^{2n+1} - (1-\varphi)^{2n+1} / \sqrt{5}\right)^{\frac{1}{2}}$$

Example $|L_5| = \left(\varphi^{11} - (1-\varphi)^{11} / \sqrt{5}\right)^{\frac{1}{2}} \approx 9.43$,

$$|L_{50}| = \left(\varphi^{101} - (1-\varphi)^{101} / \sqrt{5}\right)^{\frac{1}{2}} \approx 23{,}940{,}506{,}344.14$$

From Theorem 1, we can find the value of $|L_n|$ from the square root of F_{2n+1}. When we try to find the quotient $|L_{n+1}| / |L_n|$, we find an interesting pattern. Specifically: $|L_2|/|L_1| \approx 1.581$, $|L_3|/|L_2| \approx 1.612$ It is found that such quotients likely converge to the value Phi as n increases. This leads to the following conjecture:

Conjecture 2 $\lim (n \longrightarrow \infty) (|L_{n+1}| / |L_n|) = \varphi$

Proof: From Theorem 4.1, $|L_n| = \sqrt{F_{2n+1}}$

We get that:

$$|L_{n+1}|/|L_n| = \sqrt{F_{2(n+1)+1}} / \sqrt{F_{2n+1}} = \sqrt{F_{2n+3}/F_{2n+1}} = \sqrt{F_{2n+1} + F_{2n+2}/F_{2n+1}} = \sqrt{1 + F_{2n+2}/F_{2n+1}}$$

From fundamental knowledge, we know that

$\lim (n \longrightarrow \infty) (F_{n+1} / F_n) = \varphi$ and $1 + \varphi = \varphi^2$. Therefore

$\lim(n \to \infty)|L_{n+1}|/|L_n| = \lim(n \to \infty)\sqrt{1+F_{2n+2}/F_{2n+1}} = \sqrt{1+\varphi} = \sqrt{\varphi^2} = \varphi$

From the proof above, we can conclude that the ratio between the lengths of adjacent lines, $|L_{n+1}| / |L_n|$, converges to the Golden Ratio when $n \longrightarrow \infty$, as stated in the following theorem:

Theorem 2 $\lim (n \longrightarrow \infty) (|L_{n+1}| / |L_n|) = \varphi$

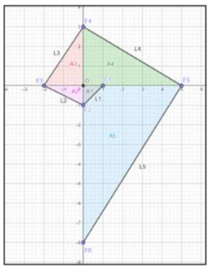

Figure 2 triangles in the Fibonacci spiral

Next, we will consider the ratio of the areas of the adjacent right-angled triangles in the Fibonacci spiral. Let A_n represent the area of the n-th triangle as shown in Figure 2. From initial observation, we find that: $A_2 / A_1 = 2.0$, $A_3 / A_2 = 3.0$, $A_4 / A_3 \approx 2.5$,...We find that the quotient is an oscillating sequence that likely converges to $1+\varphi$ as n increases. This leads to the following conjecture:

Conjecture3 $\lim (n \longrightarrow \infty)(A_{n+1}/A_n) = 1+\varphi = \varphi^2$

Proof: Since $A_n = (1/2) \times F_n \times F_{n+1}$

Therefore:

$A_{n+1}/A_n = (1/2)\times F_{n+1}\times F_{n+2}/(1/2)\times F_n \times F_{n+1} = F_{n+2}/F_n = (F_n + F_{n+1})/F_n = 1+F_{n+1}/F_n$

Therefore: $\lim (n \longrightarrow \infty) (A_{n+1} / A_n) = 1+\varphi = \varphi^2$

From the proof above, we can conclude that when $n \longrightarrow \infty$, the ratio of the area of triangle A_{n+1} to A_n converges to the square of the Golden Ratio. This can be summarized as the following theorem:

Theorem 3 $\lim (n \longrightarrow \infty) (A_{n+1} / A_n) = 1+\varphi = \varphi^2$

Next, we will consider the size of the angle that line L_n makes with line L_{n+1} when $n \longrightarrow \infty$. From exploration using GeoGebra as shown in Figure 3, it is found that the size of the angle between adjacent lines is an oscillating sequence that likely converges to 90°. This leads to the following conjecture:

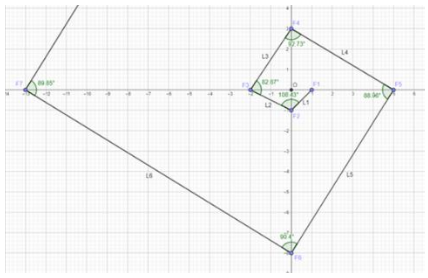

Figure 3 Angles between adjacent line

Conjecture 4 L_n is perpendicular to L_{n+1} when $n \longrightarrow \infty$.

Proof: Let mL_n be the slope of line L_n and mL_{n+1} be the slope of line L_{n+1}.

Case: L_n is in the 4th Quadrant. We have that L_n connects point $(F_n, 0)$ and $(0, -F_{n+1})$ and L_{n+1} connects point $(0, -F_{n+1})$ and $(-F_{n+2}, 0)$. Therefore:

$mL_n = -F_{n+1} - 0/0 - F_n = F_{n+1}/F_n$ and

$mL_{n+1} = 0 + F_{n+1}/-F_{n+2} - 0 = -F_{n+1}/F_{n+2}$.

So, the slope of line L_n and L_{n+1} when $n \longrightarrow \infty$ is $\lim(n \to \infty)(F_{n+1}/F_n) = \varphi$ and

$\lim(n \to \infty)(-F_{n+1}/F_{n+2}) = -\lim(n \to \infty)\left(1/(F_{n+2}/F_{n+1})\right) = -1/\varphi$

respectively. Considering the product of slopes $mL_n \times mL_{n+1}$ It yields that $\varphi \times (-1/\varphi) = -1$. That is, L_n is perpendicular to L_{n+1} when $n \longrightarrow \infty$. The cases where L_n is in

Quadrants 1, 2, and 3 can be shown in a similar manner.

From the proof above, we can conclude that when $n \longrightarrow \infty$, line L_n will be perpendicular to line L_{n+1} . This can be summarized as the following theorem:

Theorem 4 L_n is perpendicular to L_{n+1} when $n \longrightarrow \infty$.

Corollary 2 L_n is parallel to L_{n+2} when $n \longrightarrow \infty$.

Proof: From Theorem 4, we have that if $n \longrightarrow \infty$, then L_n is perpendicular to L_{n+1}, and L_{n+1} is perpendicular to L_{n+2}. Since lines that are perpendicular to the same line are parallel to each other, Therefore, it is concluded that L_n is parallel to L_{n+2}.

Conclusions

The project "Fibonacci Spiral in the Cartesian Coordinate System: Imagination from Graph to Infinite Length and Golden Ratio" began with a small question: "What would happen if we used Fibonacci numbers to define points spiraling continuously in a Cartesian coordinate system and then connected the lines in sequence?"

From that question, it became a journey of amazing discovery. The organizer studied the basic knowledge of Fibonacci numbers and geometric principles on the Cartesian plane, then created a geometric figure by drawing lines between Fibonacci points on the X and Y axes, rotating in an infinite loop. The result was not just an eye-catching geometric figure, but a mathematical phenomenon concealed with deep structures in terms of length, ratio, area, and angle. In this project, we named this figure the "Fibonacci Spiral in the Cartesian Coordinate System."

From concrete observation to formulating conjectures regarding infinity, and extending to mathematical proofs, the organizer demonstrated that the length of each line in the spiral can be expressed as the square root of a specific Fibonacci number in the sequence. Furthermore, when the numbers increase, the ratio between the lengths of adjacent lines converges to the "Golden Ratio." It was also proven that adjacent line segments in this sequence tend to be perpendicular to each other, and lines separated by two positions tend to be parallel. Additionally, the organizer developed a simple computer program capable of calculating the line lengths and the total length of the spiral, reflecting a clear integration of mathematics and technology.

Discussion

What this project clearly demonstrates is that "Mathematics is not just a subject in a textbook, but the language nature uses to design itself." From simple straight lines connecting points, we discovered the depth of infinite geometry through observation, questioning, and searching for hidden meanings in simple things. The spiral formed by Fibonacci numbers is not just connecting dots, but a representation of growth, perfect ratios, balanced relationships, and the beauty existing in the structure of infinity. The results of the proofs and the developed model, besides providing theoretical understanding, may serve as a foundation for future practical applications, such as modeling the eye of a storm or a galaxy, or designing artistic elements based on the Golden Ratio to achieve proportions that feel naturally balanced to human perception. The results of this project could potentially extend to designs requiring "mathematical precision." Most importantly, in the dimension of the learner, this project served as a comprehensive learning platform—from observation, analysis, proof, and creative thinking to using technology for visualization. Every step drove the organizer not only to understand the mathematics learned in the classroom better but to truly feel it.

One of the most amazing discoveries of this project is that when we calculate the ratio of adjacent line lengths in the Fibonacci spiral, i.e., $|L_{n+1}| / |L_n|$, we find that the value gets closer to a specific constant as $n \longrightarrow \infty$. That constant is the Golden Ratio or φ (Phi). The Golden Ratio is not just a beautiful number in mathematics but a natural phenomenon appearing in flowers, seashells, galaxies, and even in the proportions of the human body. This project reinforces the example that nature is based on mathematical principles, as stated by Galileo: "Nature is written in mathematical language" (The Galileo Project, 2024).

Another fascinating phenomenon from the findings of this project is the perpendicularity of L_n and L_{n+1}. When considering the slope of each line in the sequence, we find that when $n \longrightarrow \infty$, the product of the slopes of L_n and L_{n+1} approaches -1, which in geometry means the two lines are perpendicular. This perpendicularity is not just a simple geometric phenomenon, but a rhythm of rotation that creates a "geometrically balanced growth" perfectly. Its interest lies in the fact that even as the line lengths increase indefinitely, the angular balance is never lost. This signifies the principle of "Order in Infinity," a key property mathematicians seek in the structures they study. The results of this project are not only a theoretical mathematical success but could potentially be expanded in the future to various practical fields, including natural sciences, design, architecture, and technology.

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